A NEW APPROACH TO PERIOD ESTIMATION

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ABSTRACT

The detection and estimation of multi-periodic signals of unknown periods in white Gaussian noise is investigated. New estimates for the sub-signals (signals making up the received signal) and their periods are derived using an orthogonal subspace decomposition approach.

1. INTRODUCTION

Previous work in the period estimation area can be found in [1]-[7]. Our work extends the results of [1] to multiple period estimation. Our approach is to generate orthogonal subspaces that correspond to periods ranging from 1, to the maximum expected sub-period (P_{max}) of our signal R. Estimates of the sub-signals and their energy are obtained by taking orthogonal projections of R onto these different orthogonal subspaces. We first analyze the one period and two period estimation cases, then we will generalize the results to multiple period estimation.

2. SINGLE PERIOD ESTIMATION

We briefly review the results in [1] for the single period estimation case. Let $S = \{s_0, \ldots, s_{K_0-1}\}$ be a periodic repetition of the length P sequences $Q = \{q_0, \ldots, q_{P-1}\}$. The received signal $R = \{r_0, \ldots, r_{K_0-1}\}$, of length K_0 (K_0 is a multiple of P) then consists of S plus white, zero mean Gaussian noise $N \sim \mathcal{N}(0, \sigma^2)$

$$R = S + N$$

For any specific period P, an orthonormal basis set for the subspace of all periodic signals of period P is

$$\{\psi_k\} = \sqrt{\frac{1}{M}}\,\delta_k$$

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where $k = 0, \ldots, P-1$, $M = K_0/P$ and δ_k a $K_0 \times 1$ vector with i^{th} entry

$$\delta_k(i) = \begin{cases} 1 & i = k + lM, \text{ for integer } l \\ 0 & \text{else} \end{cases}$$

Let Ψ^P be the orthonormal matrix having $\psi_0, \ldots, \psi_{P-1}$ as column vectors

$$[\Psi^P]_{K_0 \times P} = \sqrt{\frac{1}{M}} \begin{bmatrix} 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1 \end{bmatrix}^T$$

and $\mathcal{R}(\Psi^P)$ be the range of Ψ^P. Then, $\mathcal{R}(\Psi^P)$ is the subspace of signals of period P and any other period $\bar{P}$, for which $\bar{P}$ is a factor of P. For single period estimation, the Maximum Likelihood (ML) estimate maximizes the 2-norm of $\hat{S}$, the projection of R onto $\mathcal{R}(\Psi^P)$ [1]. With $\phi_R(k)$, the autocorrelation function of R, defined as

$$\phi_R(k) = \sum_{j=0}^{K_0-1-k} r_j r_{j+k}$$

it is shown in [1] that

$$\|\hat{S}\|^2 = \frac{P}{K_0}\left[\phi_R(0) + 2\sum_{l=1}^{M-1} \phi_R(lP)\right] \quad (1)$$

The first term in (1), $\frac{P}{K_0}\phi_R(0)$, grows linearly with P. In order to eliminate some of the bias towards larger periods, in [1], it was eliminated. The proposed period estimate in [1] was

$$\hat{P} = \arg\max_P \left\{ g_{(P,R)} = \frac{2P}{K_0}\sum_{l=1}^{M-1}\phi_R(lP) \right\} \quad (2)$$

and the signal estimates $\hat{q}_i$ for $i = 0, \ldots, \hat{P}-1$ ($\hat{s}_k = \hat{q}_{k \bmod \hat{P}}$) were

$$\hat{q}_k = \frac{1}{M}\sum_{l=0}^{M-1} r_{k+lP}$$

3. EXACTLY PERIODIC SIGNALS

Let us see what happens when we apply the above algorithm to a simple signal of length 12. Let

$$R = [1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2]$$

Calculating the energy of the projections of R onto the corresponding subspaces, $\mathcal{R}(\Psi^k)$, using equation (1), we notice a few things:

1. Although the signal is of period 2, we have significant energy at period 1 (the dc value) and at period 3. This is due to the non-zero dc component and the fact that a dc signal is also periodic with period 3.
2. From equation (1), the signal has equal energy at period 2 and 4. So what period does the signal have? Is it 2 or 4? If a signal is of period P it will also be of period $2P$, $3P$, etc.

In order to clear up the above ambiguities, we introduce the following definition.

Definition 1 *We say that a signal S is of* ***exactly period*** *P if the projection of S onto $\mathcal{R}(\Psi^P)$ is non-zero and the projection of S onto $\mathcal{R}(\Psi^{\bar{P}})$ is zero for all $\bar{P} < P$.*

In our example, the received signal R is not **exactly period** 2 since the projection of R onto $\mathcal{R}(\Psi^1)$ is not zero. Similarly, the signal is not **exactly period** 4 or **exactly period** 3. A signal that would be **exactly period** 4 would be

$$R = [-1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1].$$

With our new definition, let $S_1, \ldots, S_m$ be **exactly periodic** with periods $P_1, \ldots, P_m$, respectively. The received signal R, of length K_0 (K_0, a multiple of $P_1, \ldots, P_m$) then consists of $S_1, \ldots, S_m$ plus zero mean, white Gaussian noise N

$$R = S_1 + \ldots + S_m + N$$

With an unknown variance, the ML estimator maximizes the 2-norm of the sum of the estimates of the sub-signals $\left(\hat{S}_1 + \ldots + \hat{S}_m\right)$, using estimates $\hat{\sigma}^2$ and $\hat{P}_1, \ldots, \hat{P}_m$. One way to obtain the 2-norm squared of $\left(\hat{S}_1 + \ldots + \hat{S}_m\right)$ is to find orthogonal subspaces corresponding to these signals of **exactly periods** $P_1, \ldots, P_m$. Then, projecting R onto these orthogonal subspaces we obtain estimates $\hat{S}_1, \ldots, \hat{S}_m$. Since the subspaces are orthogonal, so will be $\hat{S}_1, \ldots, \hat{S}_m$ and

$$\|\sum_i^m \hat{S}_i\|^2 = \sum_i^m \|\hat{S}_i\|^2$$

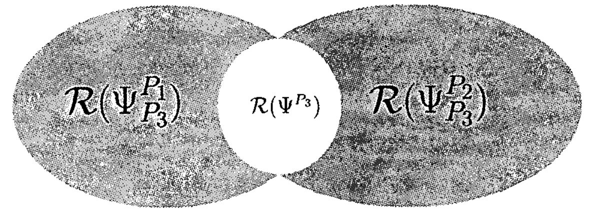

Figure 1: Subspaces $\mathcal{R}(\Psi^{P_1}_{P_3})$, $\mathcal{R}(\Psi^{P_3})$ and $\mathcal{R}(\Psi^{P_2}_{P_3})$ are mutually orthogonal. $\mathcal{R}(\Psi^{P_1}) = \mathcal{R}(\Psi^{P_1}_{P_3}) \oplus \mathcal{R}(\Psi^{P_3})$ and $\mathcal{R}(\Psi^{P_2}) = \mathcal{R}(\Psi^{P_2}_{P_3}) \oplus \mathcal{R}(\Psi^{P_3})$.

In other words, if we project R onto orthogonal subspaces corresponding to signals of **exactly period** P (with P ranging from 1 to P_{max}), the ML estimator of periods would select the m largest 2-norm projections.

4. ORTHOGONAL SUBSPACE DECOMPOSITION

Next, we characterize subspaces corresponding to signals of **exactly period** P (theorem 2). We also show that these subspaces are indeed orthogonal to each other (theorem 1). We introduce the following definition:

Definition 2 *Define $\Psi^P_{p_1,\ldots,p_m}$, with p_i divisors of P (here, we consider 1 a divisor, but not P), to be the matrix whose range is the orthogonal complement of $\mathcal{R}\left[\Psi^{p_1} \ldots \Psi^{p_m}\right]$ inside $\mathcal{R}(\Psi^P)$:*

$$\mathcal{R}(\Psi^P_{p_1,\ldots,p_m}) = \mathcal{R}(\Psi^P) \cap (\mathcal{R}\left[\Psi^{p_1} \ldots \Psi^{p_m}\right])^\perp$$

Since $\mathcal{R}(\Psi^{p_i}) \subset \mathcal{R}(\Psi^P)$, $\Psi^P_{p_1,\ldots,p_m}$ is not empty. As we show, if p_i are all the possible divisors of P (including 1 and excluding P) then $\mathcal{R}(\Psi^P_{p_1,\ldots,p_m})$ is the subspace corresponding to signals of **exactly period** P. First, we introduce the following lemma:

Lemma 1 *Given a signal R of length K_0 (K_0 a multiple of P_1 and P_2), let $\mathcal{R}(\Psi^{P_1})$ be the subspace corresponding to period P_1 and $\mathcal{R}(\Psi^{P_2})$ be the subspace corresponding to period P_2. Also, let $\mathcal{R}(\Psi^{P_3})$ be the subspace corresponding to period P_3, where P_3 is the greatest common divisor of P_1 and P_2. Then $\mathcal{R}(\Psi^{P_3})$ is the intersection of $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$. Moreover, the orthogonal complement of $\mathcal{R}(\Psi^{P_3})$ in $\mathcal{R}(\Psi^{P_1})$, $\mathcal{R}(\Psi^{P_1}_{P_3})$, is orthogonal to the orthogonal complement of $\mathcal{R}(\Psi^{P_3})$ in $\mathcal{R}(\Psi^{P_2})$, $\mathcal{R}(\Psi^{P_2}_{P_3})$. In other words, the three subspaces of Fig. 1 are mutually orthogonal.*

Proof. Assume that we have two periods P_1 and P_2 and that our received signal R is of length K_0 (K_0 a multiple of P_1 and P_2: $K_0 = P_1 M_1 = P_2 M_2$). Using the notation of section (2), $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ are two subspaces corresponding to signals of period P_1 (and any of the sub-divisors of P_1) and signals of period P_2 (and any of the sub-divisors of P_2), respectively. For the sake of clarity let $P_1 = 4$ and $P_2 = 6$. By definition Ψ^{P_1} and Ψ^{P_2} are

$$\Psi^{P_1} = \sqrt{\tfrac{1}{M_1}} \begin{bmatrix} 1000... \\ 0100... \\ 0010... \\ 0001... \end{bmatrix}^T, \Psi^{P_2} = \sqrt{\tfrac{1}{M_2}} \begin{bmatrix} 100000... \\ 010000... \\ 001000... \\ 000100... \\ 000010... \\ 000001... \end{bmatrix}^T$$

Let P_3 be the greatest common divisor of P_1 and P_2. Any signal of period P_3 is in both $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ $(\mathcal{R}(\Psi^{P_3}) \subset [\mathcal{R}(\Psi^{P_1}) \cap \mathcal{R}(\Psi^{P_1})])$. Next, any signal in $\mathcal{R}(\Psi^{P_1})$ must be of some period $\hat{P}_1$, with $\hat{P}_1$ a divisor of P_1 and any signal in $\mathcal{R}(\Psi^{P_2})$ must be of some period $\hat{P}_2$, with $\hat{P}_2$ a divisor of P_2. Therefore, any signal that's in both $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ must be of period $\hat{P}$, for which $\hat{P}$ is a divisor of both P_1 and P_2. In particular, $\hat{P}$ will be a divisor of P_3 and any signal that's in both $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ must be in $\mathcal{R}(\Psi^{P_3})$ $([\mathcal{R}(\Psi^{P_1}) \cap \mathcal{R}(\Psi^{P_1})] \subset \mathcal{R}(\Psi^{P_3}))$. Thus, the intersection of $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ is $\mathcal{R}(\Psi^{P_3})$, as shown in Fig. 1.

Next, let's find $\mathcal{R}(\Psi^{P_1}_{P_3})$, $\mathcal{R}(\Psi^{P_2}_{P_3})$ and $\mathcal{R}(\Psi^{P_3})$ such that $\mathcal{R}(\Psi^{P_1}) = \mathcal{R}(\Psi^{P_1}_{P_3}) \oplus \mathcal{R}(\Psi^{P_3})$ and $\mathcal{R}(\Psi^{P_2}) = \mathcal{R}(\Psi^{P_2}_{P_3}) \oplus \mathcal{R}(\Psi^{P_3})$. We claim that subspaces $\mathcal{R}(\Psi^{P_1}_{P_3})$, $\mathcal{R}(\Psi^{P_2}_{P_3})$ and $\mathcal{R}(\Psi^{P_3})$ are

$$\begin{aligned} \mathcal{R}(\Psi^{P_3}) &= \mathcal{R}(\Psi^{P_1}) \cap \mathcal{R}(\Psi^{P_2}) && (3) \\ &= \mathcal{R}\left[\Psi^{P_1}(\Psi^{P_1})^T\Psi^{P_2}\right] && (4) \\ &= \mathcal{R}\left[\Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}\right] && (5) \\ \mathcal{R}(\Psi^{P_1}_{P_3}) &= \mathcal{R}\left[\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}\right] && (6) \\ \mathcal{R}(\Psi^{P_2}_{P_3}) &= \mathcal{R}\left[\Psi^{P_2} - \Psi^{P_1}(\Psi^{P_1})^T\Psi^{P_2}\right] && (7) \end{aligned}$$

To prove equations (4) and (5) is a bit long and we omit the complete proof here. Intuitively equations (4) and (5) say that the intersection of the subspaces $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ can be obtained by projecting $\mathcal{R}(\Psi^{P_2})$ onto $\mathcal{R}(\Psi^{P_1})$, or the other way around. To prove equations (6) and (7) we have to show that $\mathcal{R}\left[\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1})\right]$ is the orthogonal complement of $\mathcal{R}(\Psi^{P_3})$ in $\mathcal{R}(\Psi^{P_1})$ and that $\mathcal{R}\left[\Psi^{P_2} - \Psi^{P_1}(\Psi^{P_1})^T\Psi^{P_2}\right]$ is the orthogonal complement of $\mathcal{R}(\Psi^{P_3})$ in $\mathcal{R}(\Psi^{P_2})$.

Clearly, any signal in $\mathcal{R}(\Psi^{P_1})$ can be written as a linear combination of vectors in $\mathcal{R}\left[\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}\right]$ and vectors in $\mathcal{R}(\Psi^{P_3}) = \mathcal{R}\left[\Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}\right]$; and similarly for $\mathcal{R}(\Psi^{P_2})$.

Next, we need to show:

$$\begin{aligned} \mathcal{R}\left[\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}\right] &\perp \mathcal{R}(\Psi^{P_3}) && (8) \\ \mathcal{R}\left[\Psi^{P_2} - \Psi^{P_1}(\Psi^{P_1})^T\Psi^{P_2}\right] &\perp \mathcal{R}(\Psi^{P_3}) && (9) \end{aligned}$$

Since $(\Psi^{P_2})^T \times (\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}) = 0$ the first orthogonality is proved. Similarly, we have the second orthogonality. We now have that

$$\mathcal{R}(\Psi^{P_1}_{P_3}) \perp \mathcal{R}(\Psi^{P_2}) \quad \Rightarrow \quad \mathcal{R}(\Psi^{P_1}_{P_3}) \perp \mathcal{R}(\Psi^{P_2}_{P_3}) \quad (10)$$

This proves our lemma. Q.E.D.

We are now ready to prove two theorems, which will give us the orthogonality of subspaces corresponding to **exactly period** P.

Theorem 1 *For any two specific periods P and U ($P \neq U$), let $p_1, \ldots, p_n$ and $u_1, \ldots, u_m$ be all the possible divisors of P and U respectively (here, we include 1 as a divisor). Then $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ and $\mathcal{R}(\Psi^U_{u_1,\ldots,u_m})$ are orthogonal.*

Proof. Without loss of generality, let $p_1 = u_1$ be the greatest common divisor of P and U. Then, $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n}) \subset \mathcal{R}(\Phi^P_{p_1})$ and $\mathcal{R}(\Psi^U_{u_1,\ldots,u_m}) \subset \mathcal{R}(\Psi^U_{u_1})$. By lemma 1, $\mathcal{R}(\Psi^P_{p_1})$ is orthogonal to $\mathcal{R}(\Psi^U_{u_1})$. Q.E.D.

We now prove that $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ is the subspace corresponding to signals of **exactly period** P.

Theorem 2 *Let $p_1, \ldots, p_n$ be all the possible divisors of P (here we include 1 as a divisor, but not P). Then, S is **exactly period** P if and only if $S \in \mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$.*

Proof. First, assume that $S \in \mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$. For any period $\bar{P} < P$, either $\bar{P}$ is a divisor of P or relatively prime with P. If $\bar{P}$ is a divisor of P, by definition $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ is orthogonal to $\mathcal{R}(\Psi^{\bar{P}})$. If $\bar{P}$ is relatively prime with P, then their greatest common divisor is one and from theorem 1, $\mathcal{R}(\Psi^P_1)$ is orthogonal to $\mathcal{R}(\Psi^{\bar{P}}_1)$. By definition, $\mathcal{R}(\Psi^P_1)$ is orthogonal to $\mathcal{R}(\Psi^1)$. In other words $\mathcal{R}(\Psi^P_1)$ is orthogonal to $\mathcal{R}(\Psi^{\bar{P}}_1) \oplus \mathcal{R}(\Psi^1) = \mathcal{R}(\Psi^{\bar{P}})$. With $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n}) \subset \mathcal{R}(\Psi^P_1)$, $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ is orthogonal to $\mathcal{R}(\Psi^{\bar{P}})$. In other words, the projection of $S \in \mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ onto $\mathcal{R}(\Psi^{\bar{P}})$ is zero for all $\bar{P} < P$ and S is **exactly period** P

Second, assume S is **exactly period** P. By definition, $S \in \mathcal{R}(\Psi^P)$ and $S \perp \mathcal{R}(\Psi^{\bar{P}})$, for any $\bar{P} < P$. In other words, S is in the orthogonal complement of $\mathcal{R}[\Psi^{p_1}, \ldots, \Psi^{p_n}]$ inside $\mathcal{R}(\Psi^P)$: $S \in \mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$. Q.E.D.

5. CALCULATION OF ORTHOGONAL PROJECTIONS

As we stated earlier, in section (3), we now like to compute the projection of R onto the orthogonal subspaces corresponding to **exactly period** P. In calculating those projections, we can use equation (1), without ever explicitly forming the subspaces corresponding to **exactly period** P. Again, let $p_1, \ldots, p_n$ be all the possible divisors of P (including 1 and not P). By definition, the subspace corresponding to signals of **exactly period** P is the orthogonal complement of the union of the subspaces corresponding to **exactly period** p_i, inside $\mathcal{R}(\Psi^P)$. Theorem 1 then guarantees that

$$\mathcal{R}(\Psi^P) = \text{subspace of } \textbf{exactly period } P \oplus \sum_{\oplus} \text{subspace of } \textbf{exactly period } p_i$$

Here is the algorithm for calculating our projection, using Matlab notation:

```
% Array 'g' is calculated using equation (3)
ExactlyPeriod(1)=g(1);
for i=2:sqrt(length(R)),
  fact=all_possible_factors(i);
  ExactlyPeriod(i)=g(i)-sum(ExactlyPeriod(fact));
end
```

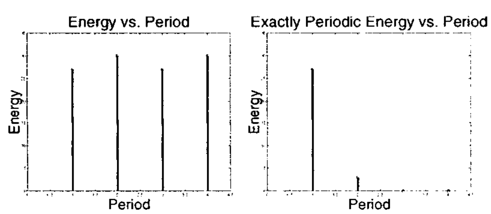

Figure 2: Plot of the energy of the projections of R onto the subspace $\mathcal{R}(\Psi^1), \mathcal{R}(\Psi^2), \mathcal{R}(\Psi^3), \mathcal{R}(\Psi^4)$ (left) and the plot of the energy of the projections of R onto the subspaces $\mathcal{R}(\Psi^1), \mathcal{R}(\Psi_1^2), \mathcal{R}(\Psi_1^3), \mathcal{R}(\Psi_{1,2}^4)$, the subspaces of **exactly period** P (right).

Applying the new transform to the signal R of section (3) we obtain the plot of Fig. 2 (right). In this transform, the signal contains a sub-signal of **exactly period** 1 and a sub-signal of **exactly period** 2. There are no sub-signals of **exactly period** 3 or **exactly period** 4. The sub-signal of **exactly period** 1 is the dc component.

6. CONCLUSION AND FUTURE WORK

We have discussed the detection and estimation of periodic signals, using an orthogonal subspace decomposition approach. Implicit in our signal model was that the signal is periodic of a length multiple of the existing periods. We need to study, in detail, the end effects for multi-periodic signals. As in [1], where the one period case was studied, the effects should be negligible.

Another issue to look at, is what happens when the periodicity of our signal changes with time. Our future work is to look at using windowing and then periodically extend the data, for improving the period estimation for this class of signals. We would like to thank Sandip Bose, at Schlumberger-Doll Research Center, for his valuable discussions and suggestions on the period estimation problem.

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